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J. Phys. A: Math. Gen. 28 (1995) 179–187. Printed in the UK

The two-singular manifold method: II. Classical Boussinesq system

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Abstract. The two-singular manifold method—a generalization of the singular manifold method of Weiss—is applied to the classical Boussinesq system, also known as the Broer–Kaup system. From the point of view of its singularity analysis, the important feature of this system is the existence of two principal families with opposite principal parts. The usual singular manifold method takes into account only one of these families at a time. Our generalization takes into account both families, and in this way we are able to derive the Lax pair and Darboux transformation—and hence the auto-Bäcklund transformation—for the classical Boussinesq system from its Painlevé analysis.

1. Introduction

The classical Boussinesq, or Broer–Kaup, system [1–3]

$$\left.\begin{aligned} E_1[U,V] &\equiv U_t + \left(V + \tfrac{1}{2}U^2\right)_x = 0 \\ E_2[U,V] &\equiv V_t + \left(a^2 U_{xx} + UV\right)_x = 0 \end{aligned}\right\} \tag{1}$$

is equivalent [2] to the scalar partial differential equation (PDE)

$$E[u] \equiv \tfrac{1}{3}a^2 u_{xxxx} - 2u_x^2 u_{xx} - \tfrac{4}{3}u_x u_{xt} - \tfrac{2}{3}u_t u_{xx} - \tfrac{1}{3}u_{tt} = 0 \tag{2}$$

where the potential u is related to U and V by

$$U = 2u_x, \quad V = -2u_t - 2u_x^2. \tag{3}$$

The system (1), or equivalently the PDE (2), is well known as a completely integrable system; it has a Lax pair and a Darboux transformation [2–5], Hirota bilinear form [6], and admits N-soliton solutions [3]. As expected, it was shown to pass the Painlevé test for PDEs [7] by Sachs [8]. Its auto-Bäcklund transformation [9], the existence of which is often taken as a definition of integrability [10], follows by a simple elimination process [11] between the Lax pair and the Darboux transformation.

An important question for a given completely integrable equation is: can we derive its Bäcklund transformation from Painlevé analysis? More importantly, can we do so without employing any 'tricks'? If we do not know how to do this for equations whose Bäcklund transformation is known, then there is little prospect of being able to do so for a new equation suspected of being integrable, for example a PDE which passes the Painlevé test.

The classical Boussinesq system is an equation for which there is as yet no satisfactory derivation of the Bäcklund transformation from Painlevé analysis. It is in fact representative

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0305-4470/95/010179+09$19.50 © 1995 IOP Publishing Ltd

of an important and large class of PDEs for which this is true. In the present paper we show how the Bäcklund transformation of the classical Boussinesq system can be obtained from the truncation procedure in Painlevé analysis.

We begin by discussing the usual truncation, i.e. the singular manifold method of Weiss [12]. This, however, only allows us to find a link with another equation, which shares with the classical Boussinesq equation (2) the property of containing Burgers equation as a factor [13]. We then consider the application of the two-singular manifold method developed by two of the present authors [14, 15]. This two-singular manifold method allows us to recover at the same time the Lax pair and the Darboux transformation, exactly like the (one-)singular manifold method does for the Korteweg–de Vries (KdV) equation. The auto-Bäcklund transformation for equation (2) follows by the usual elimination process [11]. We finish with some general conclusions on the truncation procedure in Painlevé analysis.

2. The (one-)singular manifold method

Let us briefly recall the Painlevé analysis of (2) [13]. We use the invariant formulation of this analysis [16], and so take as expansion variable χ, and also the x-primitive of χ^{-1}, $\log \Psi$ [17] (for details see the appendix). The PDE (2) then has the two expansion families

$$u = a \log \Psi + u_0 + u_1 \chi + \cdots \tag{4}$$

with a any of the two square roots of the coefficient a^2 of (2); each of these families has the indices $(-1, 0, 3, 4)$. The corresponding singularity degree of E is 4. All compatibility conditions are of course identically satisfied. These two families correspond to the invariance of (2) under $(u, t) \to (-u, -t)$, or equivalently the invariance of (1) under $(U, t) \to (-U, -t)$.

The one-family truncated expansion

$$u_{\mathrm{T}} = a \log \Psi + u_0 \tag{5}$$

exists provided that

$$C_t + \left(\tfrac{1}{2}C^2 + 2aC_x - a^2 S\right)_x = 0\,. \tag{6}$$

Such a constraint, a condition on the singular manifold $(\varphi - \varphi_0)$, is usually referred to as a 'singular manifold equation' (SME).

The natural parametric representation of the conservation law (6) is

$$C = w_x \qquad a^2 S = w_t + \tfrac{1}{2}w_x^2 + 2a w_{xx} \tag{7}$$

and the cross-derivative condition (A8) becomes

$$a^2 w_{xxxx} + 2a w_x w_{xxx} + 4a w_{xx}^2 + 2w_x^2 w_{xx} + 2(w_x w_{xt} + w_t w_{xx}) + 2a w_{xxt} + w_{tt}$$

$$\equiv \left(\partial_t + a\partial_x^2 + 2w_{xx}\right)\left(w_t + w_x^2 + a w_{xx}\right) = 0 \tag{8}$$

analogous to the factorization [13] of the classical Boussinesq equation (2)

$$E[u] \equiv -\tfrac{1}{3}\left(\partial_t - a\partial_x^2 + 2u_x\partial_x + 2u_{xx}\right)\left(u_t + u_x^2 + a u_{xx}\right) = 0\,. \tag{9}$$

The suggestion in [15] of deriving the Darboux transformation of the two-family PDE (2) from the successive application of the one-singular manifold method to each family of (2) is of no practical help here, because the Darboux transformation of the PDE (8), which has two families $w = a \log \Psi$, $3a \log \Psi$, is not easy to obtain by singularity analysis. Accordingly, we will apply the two-singular manifold method to (2) in exactly the same way as Weiss applied the one-singular manifold method to the KdV equation: this allowed him to recover both the Darboux transformation *and* the Lax pair, and we will see that the same happens with the PDE (2).

It is important to note that the truncation (5) gives a truncation for grad(u_{T}) involving only non-positive powers of χ; that is, the truncation for the original fields U and V involves only non-positive powers of χ. This is because the function Ψ is designed so that such leading-order behaviour can be identified exactly with the corresponding series in the original expansion function (singular manifold) $\varphi - \varphi_0$. In order to obtain the Lax pair for the classical Boussinesq equation we have to use a more general expansion function than χ, since χ contains information about only one singular manifold.

3. The two-singular manifold method

For the two-singular manifold method [15] we take as expansion variable a function Y, the ratio of two functions ψ_1 and ψ_2 defining two singular manifolds. This function Y then satisfies the most general Riccati system with undetermined coefficients:

$$Y_x = R_0 + R_1 Y + R_2 Y^2 \tag{10}$$

$$Y_t = S_0 + S_1 Y + S_2 Y^2 \tag{11}$$

the cross-derivative condition of which is

$$(Y_x)_t - (Y_t)_x \equiv X_0 + X_1 Y + X_2 Y^2 = 0 \tag{12}$$

where

$$X_0 \equiv R_{0,t} - S_{0,x} + R_1 S_0 - R_0 S_1 = 0 \tag{13}$$

$$X_1 \equiv R_{1,t} - S_{1,x} + 2(R_2 S_0 - R_0 S_2) = 0 \tag{14}$$

$$X_2 \equiv R_{2,t} - S_{2,x} + R_2 S_1 - R_1 S_2 = 0\,. \tag{15}$$

More generally we take components $(Y_1, Y_2, \ldots)$ satisfying a projective Riccati system.

The system (10), (11) can always be linearized by the canonical transformation $Y = \psi_1/\psi_2$ onto the essentially two-component first-order linear system

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} \frac{1}{2}R_1 & R_0 \\ -R_2 & -\frac{1}{2}R_1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{16}$$

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t = \begin{pmatrix} \frac{1}{2}S_1 & S_0 \\ -S_2 & -\frac{1}{2}S_1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{17}$$

but in general cannot be linearized onto a second-order scalar system, because this requires that either R_0 or R_2 never vanishes. The above matrix spectral problem has of course the same compatibility conditions (13), (14), and (15) as the Riccati system in Y. For PDEs having two principal families (in the sense of [18]) with opposite principal parts, the associated Darboux transformation will then in general be of the form

$$u_{\mathrm{T}} = \mathcal{D}\log\psi_1 - \mathcal{D}\log\psi_2 + v \tag{18}$$

where $\mathcal{D}$ is the singular part operator [19] and u_{T}, v are two different solutions of the PDE. The more general Riccati system (10), (11) is therefore particularly well suited to the two-manifold situation. Further discussion of the two-manifold method can be found in [15].

3.1. Application to the classical Boussinesq system

The truncation (5) is replaced with

$$u_T = a \log Y + u_0 . \tag{19}$$

From the definition of grad(Y) we see immediately that the resulting truncation for grad(u_T) extends from Y^{-1} to Y, and so those for U and V extend from Y^{-1} to Y, and from Y^{-2} to Y^2, respectively. The possibility of constructing such truncations of Painlevé expansions involving both negative and positive powers was first remarked upon in [20]. It is this together with the variable Y, satisfying a more general Riccati system than χ, that allows here the recovery of the Lax pair and the Darboux transformation for the classical Boussineq equation.

Substitution of (19) into (2) gives an expansion in Y extending from Y^{-4} to Y^4

$$E[u_T] \equiv Y^{-4} \sum_{j=0}^{8} E_j Y^j = 0 \tag{20}$$

generating nine determining equations, together with the three cross-derivative conditions (13)–(15). Among these, E_0 and E_8 are identically zero since in (19) the leading-order coefficient has already been chosen, and E_j, E_{8-j} for the Fuchs indices $j = 3, 4$ are differential consequences of (E_1, E_2), (E_7, E_6) respectively. This then leaves us with four determining equations,

$$\frac{E_1}{R_0^3} \equiv 2a^2 \left(a \frac{R_{0,x}}{R_0} + a R_1 + \frac{S_0}{R_0} + 2u_{0,x} \right) = 0 \tag{21}$$

$$\begin{aligned} \frac{3E_2}{aR_0^2} \equiv\ & a \left(2aR_1 - 2\frac{S_0}{R_0} - 12u_{0,x} \right) \frac{R_{0,x}}{R_0} - 3a^2 \frac{R_{0,x}^2}{R_0^2} \\ & -4a^2 \frac{R_{0,xx}}{R_0} - 2a^2 R_{1,x} - 4a \frac{S_{0,x}}{R_0} + 24aR_1 u_{0,x} \\ & +2u_{0,t} + 6u_{0,x}^2 - 6a u_{0,xx} + 4\frac{S_0}{R_0} u_{0,x} + 11a^2 R_1^2 \\ & +4a^2 R_0 R_2 + 10aR_1 \frac{S_0}{R_0} + \left(\frac{S_0}{R_0} \right)^2 + 2aS_1 = 0 \end{aligned} \tag{22}$$

$$\begin{aligned} \frac{3E_6}{aR_2^2} \equiv\ & a \left(2aR_1 - 2\frac{S_2}{R_2} - 12u_{0,x} \right) \frac{R_{2,x}}{R_2} + 3a^2 \frac{R_{2,x}^2}{R_2^2} \\ & +4a^2 \frac{R_{2,xx}}{R_2} - 2a^2 R_{1,x} - 4a \frac{S_{2,x}}{R_2} - 24aR_1 u_{0,x} \\ & -2u_{0,t} - 6u_{0,x}^2 - 6a u_{0,xx} - 4\frac{S_2}{R_2} u_{0,x} - 11a^2 R_1^2 \\ & -4a^2 R_0 R_2 - 10aR_1 \frac{S_2}{R_2} - \left(\frac{S_2}{R_2} \right)^2 - 2aS_1 = 0 \end{aligned} \tag{23}$$

$$\frac{E_7}{R_2^3} \equiv 2a^2 \left(a \frac{R_{2,x}}{R_2} - a R_1 - \frac{S_2}{R_2} - 2u_{0,x} \right) = 0 \tag{24}$$

where t-derivatives of R_0 and R_2 have been eliminated using the cross-derivative conditions (13) and (15).

Assuming $R_0 R_2 \neq 0$, these four equations are equivalent, modulo the cross-derivative conditions (13)–(15), to

$$\frac{S_0}{R_0} = -2u_{0,x} - a((\log R_0)_x + R_1) \tag{25}$$

$$\frac{S_2}{R_2} = -2u_{0,x} + a((\log R_2)_x - R_1) \tag{26}$$

$$\frac{E_2}{R_0^2} + \frac{E_6}{R_2^2} \equiv \tfrac{4}{3}a^2(aR_1 + u_{0,x})_x = 0 \tag{27}$$

$$\frac{E_2}{R_0^2} - \frac{E_6}{R_2^2} \equiv \tfrac{4}{3}a(aR_1 + u_{0,x})_t + \tfrac{8}{3}a^2(aR_1 + u_{0,x})(aR_1 + u_{0,x})_x = 0\,. \tag{28}$$

This introduces the spectral parameter as an arbitrary constant of integration and leads to the solution for R_1, S_0, S_1, and S_2

$$R_1 = \lambda - a^{-1}u_{0,x} \tag{29}$$

$$S_0 = -a\lambda R_0 - aR_{0,x} - R_0 u_{0,x} \tag{30}$$

$$S_1 = -a\lambda^2 - 2aR_0R_2 - a^{-1}u_{0,t}, \tag{31}$$

$$S_2 = -a\lambda R_2 + aR_{2,x} - R_2 u_{0,x} \tag{32}$$

with the two remaining nonidentically zero cross-derivative conditions

$$a\frac{X_0}{R_0} \equiv (a\partial_x^2 + \partial_t)(u_0 + a\log R_0) + 2a^2R_0R_2 + \left(u_{0,x} + a\frac{R_{0,x}}{R_0}\right)^2 = 0 \tag{33}$$

$$a\frac{X_2}{R_2} \equiv (a\partial_x^2 - \partial_t)(u_0 - a\log R_2) - 2a^2R_0R_2 - \left(u_{0,x} - a\frac{R_{2,x}}{R_2}\right)^2 = 0\,. \tag{34}$$

Elimination of R_0 or R_2 leads to the conditions

$$X_0 \equiv \frac{3}{2a^2R_2}E[u_0 - a\log R_2] = 0 \qquad X_2 \equiv \frac{3}{2a^2R_0}E[u_0 + a\log R_0] = 0 \tag{35}$$

expressing that a second solution v to the classical Boussinesq system has been obtained. Since R_0 and R_2 are exchanged under the permutation of (ψ_1, ψ_2), this provides for the Riccati pseudopotential $y = R_2 Y$

$$y_x = r_0 + r_1 y + r_2 y^2 \tag{36}$$

$$y_t = s_0 + s_1 y + s_2 y^2 \tag{37}$$

the *unique* solution

$$r_0 = \frac{1}{2a^2}(-v_t - v_x^2 + av_{xx}) \tag{38}$$

$$r_1 = \lambda - \frac{1}{a}v_x \tag{39}$$

$$r_2 = 1 \tag{40}$$

$$s_0 = \frac{1}{2a^2}(a\lambda + v_x)(v_t + v_x^2) + \frac{1}{2a}(v_{xt} + v_x v_{xx}) - \tfrac{1}{2}\lambda v_{xx} - \tfrac{1}{2}v_{xxx} \tag{41}$$

$$s_1 = -a\lambda^2 + \frac{1}{a}v_x^2 - v_{xx} \tag{42}$$

$$s_2 = -a\lambda - v_x \tag{43}$$

with the conservative form for the second equation (37)

$$y_t = \left(-(v_x + a\lambda)y + \frac{1}{2a}(v_t + v_x^2 - av_{xx})\right)_x . \tag{44}$$

The transformation $y = \psi_1/\psi_2$ gives a matrix Lax pair (16), (17), with $R_i = r_i$ and $S_j = s_j$. The truncation u_T gives the corresponding Darboux transformation

$$u_T = a \log Y + u_0 = a \log y + v = a \log \psi_1 - a \log \psi_2 + v \tag{45}$$

which is exactly of the form (18).

The two-singular manifold method thus gives both the Lax pair and the Darboux transformation in the case of the Broer–Kaup system. This situation is identical to that of the (one-)singular manifold method of Weiss when applied to the Korteweg and de Vries equation [7, 15]: the Darboux transformation is not known before the truncation procedure and is a result of the method. Thus, in some sense, the Broer–Kaup system is as elementary among the two-family equations as the KdV equation is among the one-family ones.

Elimination [11] of y between the Riccati pseudopotential (36), (37) and the Darboux transformation (45) gives the auto-Bäcklund transformation for the classical Boussinesq system [9]

$$(u - v)_x = a\lambda - v_x + e^{(u-v)/a} - \frac{1}{2a}e^{(v-u)/a}(v_t + v_x^2 - av_{xx}) \tag{46}$$

$$\begin{aligned}(u - v)_t = &-a^2\lambda^2 + v_x^2 - av_{xx} - ae^{(u-v)/a}(a\lambda + v_x)\\ &+\frac{1}{2a}e^{(v-u)/a}(a\lambda + v_x + a\partial_x)(v_t + v_x^2 - av_{xx})\end{aligned} \tag{47}$$

where we have written u for u_T. (For this particular example, y can be easily eliminated since the singular part operator $\mathcal{D}$ is invertible: $y = e^{(u-v)/a}$.)

The Riccati system (36), (37) can also be linearized (because r_2 is a never-vanishing constant) onto a scalar Lax pair by

$$y = -\frac{\eta_x}{\eta} \tag{48}$$

$$\eta_{xx} = \left(\lambda - \frac{1}{2a}\overline{U}\right)\eta_x - \frac{1}{4a^2}\left(\overline{V} + a\overline{U}_x\right)\eta \tag{49}$$

$$\eta_t = -a\left(\lambda + \frac{1}{2a}\overline{U}\right)\eta_x + \frac{1}{4a}(2a^2\lambda^2 + \overline{V} + a\overline{U}_x)\eta \tag{50}$$

the spatial part of which is linear in λ and the physical fields $\overline{U}, \overline{V}$ linked to v by

$$\overline{U} = 2v_x \qquad \overline{V} = -2v_t - 2v_x^2 . \tag{51}$$

In (50) the arbitrary gauge coming from the integration with respect to x in (44) has been set equal to the constant $a\lambda^2/2$. The cross-derivative condition of the pair (49), (50) then reads

$$(\eta_{xx})_t - (\eta_t)_{xx} = -\frac{1}{2a}E_1[\overline{U}, \overline{V}]\eta_x - \frac{1}{4a^2}\left\{E_2[\overline{U}, \overline{V}] + a\left(E_1[\overline{U}, \overline{V}]\right)_x\right\}\eta = 0 \tag{52}$$

and so is satisfied iff $\overline{U}$, $\overline{V}$ satisfy the classical Boussinesq system (1).

If one linearizes the Riccati system (36), (37) so as to cancel the coefficient of η_x in the spatial part (49), one obtains

$$y = -\frac{\psi_x}{\psi} + \frac{1}{4a}(\overline{U} - 2a\lambda) \tag{53}$$

$$\psi_{xx} = \frac{1}{16a^2}\left[(\overline{U} - 2a\lambda)^2 - 4\overline{V}\right]\psi \tag{54}$$

$$\psi_t = -\tfrac{1}{2}\left(\overline{U} + 2a\lambda\right)\psi_x + \tfrac{1}{4}\overline{U}_x\psi\,. \tag{55}$$

This is the Lax pair for the classical Boussinesq equation given by Kaup [2]; the nonlinearity in λ and U of the spatial part (54) of this Lax pair is due to the shift incorporated in (53) which removes terms in ψ_x from (54).

3.2. Remarks

There has recently been another attempt [21] to obtain the Lax pair for the classical Boussinesq equation from its Painlevé analysis. Using arguments based on the Hirota bilinear form of the classical Boussinesq equation [6], a solution was sought depending explicitly on two singular manifolds. However, these authors were not able to obtain the Lax pair with a spectral parameter λ — the latter had to be introduced afterwards using the Galilean invariance of the classical Boussinesq system. Moreover, the manipulation of expansions involving two (or more) singular manifolds is complicated, and was not handled systematically in [21].

We stress therefore the following advantages of the approach adopted herein:

- We use a single expansion variable Y.
- After substitution of the appropriate expansion, coefficients of each power of Y are set to zero independently.
- The spectral parameter λ is introduced by the process of solving the determining equations for the coefficients of the appropriate (projective) Riccati system.
- The singularity structure dictates the required linearization of the Riccati system.

These points are of course equally applicable to both the singular manifold method of Weiss [12, 22] and to the two-singular manifold method used here.

Finally, we remark that the idea of introducing two entire functions, originally due to Painlevé [23] for ordinary differential equations, was first introduced for PDEs by Hirota [24, 25]. The bilinear and trilinear approaches for the classical Boussinesq system, and its occurence as a reduction of first modified KP, are discussed in [26–29].

4. Conclusions

We have applied the two-singular manifold method to the classical Boussinesq equation to derive its Lax pair and Darboux transformation from Painlevé analysis. The auto-Bäcklund transformation then follows immediately, by elimination of the variable y between the two Riccati pseudopotential equations and the Darboux transformation.

Acknowledgments

This work was partially funded by Grant T.93.4 from Sous-commission des échanges scientifiques entre la France et le Royaume de Belgique (Wetenschappelijke uitwisseling:

Vlaamse Gemeenschap–Frankrijk). AP would like to thank Professor Reignier for an invitation to Dienst TENA in January 1993, at which time this work was initiated. MM acknowledges financial support from the NFWO and from the project IUAP III funded by the Belgian Government.

Appendix. The invariant analysis

The invariant analysis [16] uses as expansion variable a function χ given in terms of the singular manifold $\varphi - \varphi_0$ by

$$\chi = \left(\frac{\varphi_x}{\varphi - \varphi_0} - \frac{\varphi_{xx}}{2\varphi_x}\right)^{-1} \tag{A1}$$

and also the x-primitive of χ^{-1}, $\log \Psi$ [17], where

$$\Psi = (\varphi - \varphi_0)\varphi_x^{-1/2}. \tag{A2}$$

These functions have gradients given by

$$\chi_x = 1 + \frac{1}{2} S \chi^2 \tag{A3}$$

$$\chi_t = -C + C_x \chi - \tfrac{1}{2}(C_{xx} + CS)\chi^2 \tag{A4}$$

$$(\log \Psi)_x = \chi^{-1} \tag{A5}$$

$$(\log \Psi)_t = -C\chi^{-1} + \tfrac{1}{2} C_x \tag{A6}$$

where

$$S = \left(\frac{\varphi_{xx}}{\varphi_x}\right)_x - \tfrac{1}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2 \qquad C = -\frac{\varphi_t}{\varphi_x}. \tag{A7}$$

The cross-derivative condition on grad(χ) is identical to that on grad($\log \Psi$);

$$S_t + C_{xxx} + 2C_x S + C S_x = 0 \tag{A8}$$

and is identically satisfied in terms of φ.

This invariant analysis builds in a re-summation of the original WTC Painlevé expansion, and has the effect of greatly shortening the expressions for the coefficients of the expansion.

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